ROBUST RATE-MAXIMIZATION GAME UNDER BOUNDED CHANNEL UNCERTAINTY

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ABSTRACT

The problem of decentralized power allocation for competitive rate maximization in a frequency-selective Gaussian interference channel is considered. In the absence of perfect knowledge of channel state information (CSI), a distribution-free robust game is formulated. A robust-optimization equilibrium (RE) is proposed where each player formulates a best response to the worst-case interference. The conditions for existence, uniqueness and convergence of the RE are derived. It is shown that the convergence reduces as the uncertainty increases. Simulations show an interesting phenomenon where the proposed RE moves closer to a Pareto-optimal solution as the CSI uncertainty bound increases, when compared to the classical Nash equilibrium under perfect CSI. Thus, the robust-optimization equilibrium successfully counters bounded channel uncertainty and increases system sum-rate due to users being more conservative about causing interference to other users.

Index Terms— Rate maximization, robust games, waterfilling, decentralized power control, Gaussian interference channel

1. INTRODUCTION

The problem of competitive rate maximization is an important signal-processing problem for power-constrained multi-user systems. It involves solving the power control problem for mutually interfering users operating across multiple frequencies. The classical approach to rate maximization has been finding globally optimal solutions based on waterfilling [1]. However, the major drawback of this approach is that these solutions require centralized control. These solutions are inherently unstable in a competitive multi-user scenario, since a gain in performance for one user may result in a loss of performance for others. Instead, a distributed game-theoretic approach is desirable and is being increasingly considered only over the past decade.

The seminal work on competitive rate maximization [2] uses a game-theoretic approach to design a decentralized algorithm for two-user dynamic power control. This work proposed a sequential iterative waterfilling algorithm for reaching the Nash equilibrium (NE) in a distributed manner. A Nash equilibrium of the rate-maximization game implies that given that the power allocations of other users is constant, no user can further increase the achieved information rate unilaterally. However, this work and others extending this work such as [3], [4] and [5] all assume perfect channel state information. This is a very strong requirement and generally cannot be met by practical wireless systems.

The traditional game-theoretic solution for systems with imperfect information is the Bayesian game model [6] which uses a probabilistic approach to model the uncertainty in the system. However, a Bayesian approach is often intractable and the results strongly depend on the nature of the probability distribution functions.

The issue of bounded uncertainty in specific distributed optimization problems in communication networks has been investigated in [7] where techniques to define the uncertainty set such that they can be solved distributively by robust optimization solutions are presented. In [8], incomplete-information finite games have been modelled as a distribution-free *robust game* where the players use a robust optimization approach to counter bounded payoff uncertainty. This robust game model also introduced a distribution-free equilibrium concept called the *robust-optimization equilibrium*. However, the results in [8] for the robust game model are limited to finite games, which is not applicable here.

We present a distribution-free robust game formulation for the rate-maximization game where non-cooperative users formulate best responses to worst-case interference to counter channel uncertainty. We analyse the robust-optimization equilibrium (RE) for this game and derive sufficient conditions for its existence and uniqueness. We propose a decentralized algorithm where the players use an iterative waterfilling solution to converge to the robust-optimization equilibrium under certain sufficient conditions. Compared with the Nash equilibrium for the rate maximization game under perfect CSI [5], we show that the conditions are stronger under channel uncertainty. We compare the global efficiency of the RE and the NE through simulations and show that RE has higher efficiency due to users being more conservative about causing interference under uncertainty which encourages better partitioning of the frequencies among the users.

Notations used: The expectation operator is denoted by

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E$\{\cdot\}$. The quantity $[\mathbf{A}]_{ij}$ refers to the (i,j)-th element of $\mathbf{A}$. $\mathbb{R}_+^{m\times n}$ is the set of $m \times n$ matrices with real non-negative elements. The spectral radius (largest eigenvalue) of matrix $\mathbf{A}$ is denoted by $\rho(\mathbf{A})$ [9]. The operation $[x]_a^b$ is defined as $[x]_a^b = a$ if $x \le a$; x if $a < x < b$; b if $x \ge b$.

2. SYSTEM MODEL

We consider a system similar to the one in [5], which is a frequency-selective Gaussian interference channel with N frequencies, composed of Q SISO links. $\Omega \triangleq \{1,\ldots,Q\}$ is the set of the Q players (i.e. SISO links). The quantity $H_{rq}(k)$ denotes the normalized frequency response of the k-th frequency bin of the channel between source r and destination q. The variance of the zero-mean circularly symmetric complex Gaussian noise at receiver q in the frequency bin k is denoted by $\bar{\sigma}_q^2(k)$. Let $\sigma_q^2(k) \triangleq \bar{\sigma}_q^2(k)/|H_{qq}(k)|^2$ and the total transmit power of user q be NP_q. Let the vector $\mathbf{s}_q \triangleq [s_q(1)\ s_q(2)\ldots s_q(N)]$ be the N symbols transmitted by user q on the N frequency bins and $p_q(k) \triangleq \mathrm{E}\{|s_q(k)|^2\}/P_q$ be the power allocated to the k-th frequency bin by user q and $\mathbf{p}_q \triangleq [p_q(1)\ p_q(2)\ldots p_q(N)]$ is the power allocation vector. The power allocation vector of each user q has two constraints: (a) Maximum total transmit power for each user, $\mathrm{E}\{||\mathbf{s}_q||_2^2\} \le NP_q$; (b) Spectral mask constraints, $p_q(k) \le p_q^{max}(k)$. The power allocation vectors are public information, i.e. known to all users.

Each receiver estimates the channel between itself and all the transmitters, which is private information. This estimate is assumed to have a bounded uncertainty of unknown distribution. For simplicity and tractability, the uncertainty in the channel state information in each frequency is deterministically modelled under an ellipsoid approximation [10] as

$$\mathcal{F}_q = \left\{ F_{rq}(k) + \Delta F_{rq,k} \ :\ \sum_{r\neq q} |\Delta F_{rq,k}|^2 \le \epsilon_q^2 \right\}, \quad (1)$$

for $k = 1,\ldots,N$, where $\epsilon_q \ge 0\ \forall\ q \in \Omega$ is the degree of uncertainty and

$$F_{rq}(k) \triangleq \frac{|H_{rq}(k)|^2}{|H_{qq}(k)|^2}, \quad (2)$$

with $F_{rq}(k)$ being the nominal value. We can consider uncertainty in $F_{rq}(k)$ instead of $H_{rq}(k)$ because a bounded uncertainty in $F_{rq}(k)$ and $H_{rq}(k)$ are equivalent, but with different bounds.

3. ROBUST RATE-MAXIMIZATION GAME

According to the robust game model [8], each player formulates a best response as the solution of a robust (worst-case) optimization problem for the uncertainty in the payoff function (information rate), given the other players' strategies. If all the players know that everyone else is using the robust optimization approach to the payoff uncertainty, they would then be able to mutually predict each other's behaviour. The robust game $\mathscr{G}_{\text{rob}}$ where each player q formulates a worst-case robust optimization problem can be written as, $\forall\ q \in \Omega$,

$$\begin{aligned} &\max_{\mathbf{p}_q}\ \min_{F_{rq}}\ \sum_{k=1}^{N} \log\left(1 + \frac{p_q(k)}{\sigma_q^2(k) + \sum_{r\neq q} F_{rq}(k) p_r(k)}\right) \\ &\text{s.t.}\quad F_{rq} \in \mathcal{F}_q, \quad \mathbf{p}_q \in \mathcal{P}_q, \end{aligned} \quad (3)$$

where $\mathcal{F}_q$ is the uncertainty set which is modelled under ellipsoid approximation as shown in (1) and $\mathcal{P}_q$ is the set of admissible strategies of user q, which is defined as

$$\begin{aligned} \mathcal{P}_q \triangleq \Big\{ \mathbf{p}_q \in \mathbb{R}^N : &\frac{1}{N}\sum_{k=1}^{N} p_q(k) = 1, \\ &0 \le p_q(k) \le p_q^{max}(k), \quad k = 1,\ldots,N \Big\}. \end{aligned} \quad (4)$$

The optimization problem in (3) using uncertainty sets is equivalent to the form represented by protection functions [7]: $\forall\ q \in \Omega$,

$$\begin{aligned} &\max_{\mathbf{p}_q}\ \min_{\Delta F_{rq,k}}\ \sum_{k=1}^{N} \log\Big(1 + \frac{p_q(k)}{\sigma_q^2(k) + \sum_{r\neq q} (F_{rq}(k) + \Delta F_{rq,k}) p_r(k)}\Big) \\ &\text{s.t.}\quad \sum_{r\neq q} |\Delta F_{rq,k}|^2 \le \epsilon_q^2, \quad \mathbf{p}_q \in \mathcal{P}_q. \end{aligned} \quad (5)$$

From the Cauchy-Schwarz inequality [9], we get

$$\sum_{r\neq q} \Delta F_{rq} p_r(k) \le \epsilon_q \sqrt{\sum_{r\neq q} p_r^2(k)}. \quad (6)$$

Using (6), we get the robust game $\mathscr{G}_{\text{rob}}$ as, $\forall\ q \in \Omega$,

$$\begin{aligned} &\max_{\mathbf{p}_q} \sum_{k=1}^{N} \log\Big(1 + \frac{p_q(k)}{\sigma_q^2(k) + \sum_{r\neq q} F_{rq}(k) p_r(k) + \epsilon_q \sqrt{\sum_{r\neq q} p_r^2(k)}}\Big) \\ &\text{s.t.}\quad \mathbf{p}_q \in \mathcal{P}_q. \end{aligned} \quad (7)$$

4. ROBUST-OPTIMIZATION EQUILIBRIUM

We now present the robust waterfilling solution to the optimization problem in (7) in the following theorem:

Theorem 1. *The solution to the robust optimization problem of user q in* (7) *for a given set of power allocations of other users* $\mathbf{p}_{-q} \triangleq \{\mathbf{p}_1,\ldots,\mathbf{p}_{q-1},\mathbf{p}_{q+1},\ldots,\mathbf{p}_Q\}$ *is given by*

$$\mathbf{p}_q^\star = \mathrm{RWF_q}(\mathbf{p}_{-q}),$$

where the robust waterfilling operator $\left[\mathrm{RWF_q}(\mathbf{p}_{-q})\right]_k$ *is defined as*

$$\left[\mu_q - \sigma_q^2(k) - \sum_{r\neq q} F_{rq}(k) p_r(k) - \epsilon_q \sqrt{\sum_{r\neq q} p_r^2(k)}\right]_0^{p_q^{max}(k)}, \quad (8)$$

for $k = 1, \ldots, N$, where μ_q is chosen to satisfy the power constraint $(1/N)\sum_{k=1}^{N} p_q^{\star}(k) = 1$.

The robust waterfilling operation for each user is a distributed worst-case optimization under bounded channel uncertainties. Compared with the original waterfilling operation in [5] under perfect CSI (i.e. $\epsilon_q \equiv 0$), we see that an additional term has appeared in (8) for $\epsilon_q > 0$.

Intuitively speaking, this term acts as a penalty for allocating power to frequencies having a large product of uncertainty bound and norm of the powers of the other players currently transmitting in those frequencies. This is because the users assume the worst-case interference from other users and are thus conservative about allocating power to such channels.

The solution to the game $\mathscr{G}_{\text{rob}}$ is the robust-optimization equilibrium (RE). At any RE of this game, the optimum action profile of the players $\{\mathbf{p}_q^{\star}\}_{q\in\Omega}$ must satisfy the following set of simultaneous waterfilling equations: $\quad \forall q \in \Omega$,

$$\mathbf{p}_q^{\star} = \mathrm{RWF_q}(\mathbf{p}_{-q}^{\star}). \tag{9}$$

This equilibrium can be computed using a modified asynchronous iterative waterfilling algorithm [5] which uses the robust waterfilling operator $\mathrm{RWF_q}(\cdot)$. It can easily be verified that the RE reduces to the Nash equilibrium of the system [5] when there is no uncertainty in the system. In Section 5, we compare the global efficiency of the robust-optimization equilibrium and Nash equilibrium through simulations and show that the RE has a higher efficiency due to a penalty for interference which encourages better partitioning of the frequency space among the users.

The non-negative matrices $\mathbf{E}$ and $\mathbf{S}^{max} \in \mathbb{R}_+^{Q\times Q}$ are defined as

$$[\mathbf{E}]_{qr} \triangleq \begin{cases} \epsilon_q, & \text{if } r \neq q, \\ 0, & \text{otherwise,} \end{cases} \tag{10}$$

and

$$[\mathbf{S}^{max}]_{qr} \triangleq \begin{cases} \max\limits_{k\in\mathcal{D}_q\cap\mathcal{D}_r} F_{rq}(k), & \text{if } r \neq q, \\ 0, & \text{otherwise,} \end{cases} \tag{11}$$

where $\mathcal{D}_q$ is any subset of $\{1,\ldots,N\}$ such that $\mathcal{D}_q^{min} \subseteq \mathcal{D}_q \subseteq \{1,\ldots,N\}$. $\mathcal{D}_q^{min}$ is defined as [5] $\mathcal{D}_q^{min} \triangleq \{k \in \{1,\ldots,N\} : \exists\, \mathbf{p}_{-q} \in \mathcal{P}_{-q}$ such that $\left[\mathrm{RWF_q}(\mathbf{p}_{-q})\right]_k \neq 0\}$. The sufficient condition for existence and uniqueness of the RE of the game $\mathscr{G}_{\text{rob}}$ is given by the following theorem:

Theorem 2. *Game $\mathscr{G}_{rob}$ has at least one robust-optimization equilibrium for any set of channel matrices and transmit powers of the users. Furthermore, the robust-optimization equilibrium is unique and the asynchronous iterative waterfilling algorithm based on the robust waterfilling operator $\mathrm{RWF_q}(\cdot)$ used to reach the robust-optimization equilibrium will converge if*

$$\rho(\mathbf{S}^{max}) < 1 - \rho(\mathbf{E}), \tag{12}$$

where $\mathbf{E}$ and $\mathbf{S}$ are defined in (10) *and* (11) *respectively.*

We can easily see that this condition reduces to condition (C1) in [5] as expected in the absence of uncertainty, i.e. when $\epsilon_q \equiv 0$. Since $\rho(\mathbf{E}) \geq 0$, the probability that a given system converges to the RE decreases as the degree of uncertainty increases.

Corollary 1. *When the degrees of uncertainty of all the Q users are equal to ϵ, the sufficiency condition in* (12) *for the robust-optimization equilibrium of the game $\mathscr{G}_{rob}$ can be written as*

$$\rho(\mathbf{S}^{max}) < 1 - \epsilon(Q-1). \tag{13}$$

This expression explicitly shows how the degree of uncertainty and the number of users in the system affect the convergence to the equilibrium using an iterative waterfilling algorithm. For a fixed degree of uncertainty, as the number of users in the system increases, there is a larger amount of uncertain information in the system. Hence, the probability that a given system will converge to the RE will decrease as the number of users in the system increases.

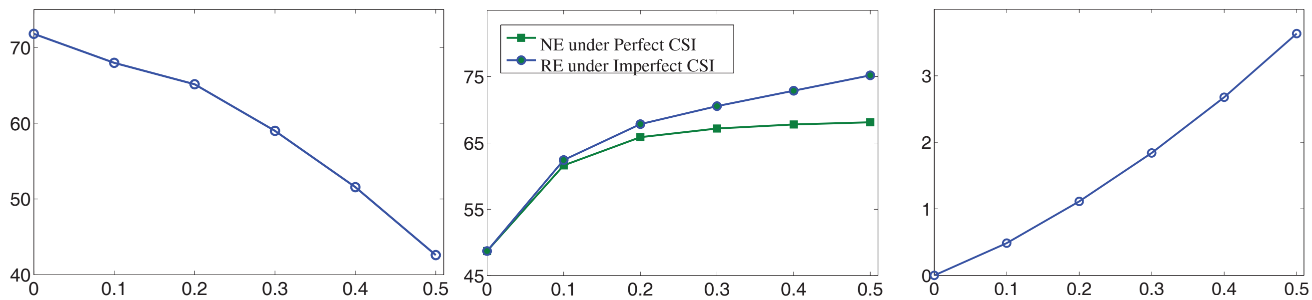
5. NUMERICAL RESULTS

In this section, we present some simulation results to study the impact of channel uncertainty on the RE by comparing it with the ideal scenario of NE under perfect CSI. The simulation parameters are presented in Figure 1. We study three important aspects, namely the percentage of convergence, the total information rate observed at the receivers and the additional number of frequencies with zero power allocations at the RE when compared to the NE against the percentage of uncertainty δ.

In Figure 1a, we observe that as the uncertainty δ increases, the fraction of trials that converge under the RE decreases. This is as expected from Corollary 1, which defines the relation between the convergence conditions and uncertainty. As the uncertainty increases, the worst-case interference increases and thus convergence does not occur. In contrast, in Figure 1b we see that the total rate increases with uncertainty.

In Figure 1b, we see that the total information rate at the NE under perfect CSI is less than the rate at the RE under imperfect CSI and that the gap in performance increases as the uncertainty increases. This may seem surprising at first as the NE has access to better information than the RE. However, in a game-theoretic situation, better information does not necessarily mean better performance. Furthermore, the NE is well-known to be inefficient in a global sense of total rate [11]. Under imperfect CSI, the power allocation using the RE[1] in (8) and (9) has higher total information rate as uncertainty increases. This is because the users are more cautious about using frequencies with significant interference, thus reducing

[1](8) and (9) are in terms of absolute uncertainty ϵ while the simulations use relative uncertainty δ. They are equivalent to one another.



(a) % of converged runs for RE vs. uncertainty δ. (b) Total rate observed at rx. vs. uncertainty δ. (c) Extra no. of zero alloc. for RE vs. uncertainty δ.

Fig. 1: Simulation results for a system with $\bar{\sigma} = 0.1$, $Q = 5$ users and $N = 16$ frequencies over 10000 runs. Channel gains $H_{rq}(k) \sim \mathcal{CN}(0, 1)$ for $r \neq q$, $H_{qq}(k) \sim \mathcal{CN}(0, 2.25)$. Channel uncertainty model: nominal value $F_{rq}(k) = F_{rq}^{\text{true}}(k)(1 + e_{rq}(k))$ with $e_{rq}(k) \sim \mathcal{U}(-\frac{\delta}{2}, \frac{\delta}{2})$, $\delta < 1$. Figs (b) and (c) are averaged over trials where the RE converges. In Fig (b), both rates are observed through the true channel values $F_{rq}^{\text{true}}(k)$. The NE scheme allocates power using $F_{rq}^{\text{true}}(k)$, while the RE scheme uses $F_{rq}(k)$.

the total amount of interference in the system. Note that, for fairness, we have averaged the rates over the same trials for the RE and NE. The NE always converges when the RE converges (from [5, Theorem 1] and Theorem 2), however the converse is not always true. The increase in the NE rate curve with δ is due to the bias in averaging only over the runs where the RE converges.

In Figure 1c, we observe that, for the average used, the total number of channels across all users that have zero power allocation at the RE increases when compared with the NE. This implies that the users are using smaller number of frequencies, which demonstrates the better partitioning of the frequency space among the users to reduce interference. Hence, this leads to the higher information rates observed in Figure 1b. The simulation results suggest that schemes similar to the RE can move closer to Pareto optimality for rate-maximization games.

6. CONCLUSIONS

This paper presented a novel approach for rate-maximization games under bounded channel state information uncertainty. We introduced a novel distribution-free robust game formulation for the rate-maximization game. We presented the robust-optimization equilibrium for this game and conditions for its existence, uniqueness and convergence. Our simulation results indicated an interesting effect of such a robust formulation retaining a decentralized structure, where the equilibrium tends to move closer to the Pareto optimal solution. This framework can be extended to MIMO rate-maximization games, cognitive radio and other non-cooperative games.

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